

# Relation between Dimension and Angular Momentum for Radially Symmetric Potential in $N$ -dimensional Space

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## Abstract

It is proved that when solving Schroedinger equations for radially symmetric potentials the effect of higher dimensions on the radial wave function is equivalent to the effect of higher angular momenta in lower dimensional cases. This result is applied to giving solutions for several radially symmetric potentials in  $N$ -dimension.

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There are more and more physical problems related to dimensions higher than 3, which have attracted much attention recently[1]. In the study of cosmology, group theory, many body problem, supersymmetry, etc. multi-dimensional solutions are often required. To solve the basic equation in quantum mechanics, Schroedinger equation, for multi-dimensional problems serves very well as the starting point for general discussions in any multi-dimensional quantum problems.

Among the solutions of Schroedinger equation, the problems with radially symmetric potential is of more interests. In fact, only the solutions for a few radially symmetric potentials in 1-, 2- and 3-dimensions are known. Much efforts have been paid to obtain the solutions or to discuss their properties for higher dimensional problems[1,2,3]. The purpose of this paper is to show a very simple way to relate the solutions in 2- and 3- dimensional problems to any higher dimensional cases for radially symmetric potentials. The basic idea is that the dimension and the angular momentum are related in a very simple way for radially symmetric potentials. Based on this relation, the solutions with lower angular momentum for higher dimensional problems can be expressed by solutions with higher angular momentum for lower dimensional problems, specially by 2- and 3-dimensional solutions.

In section 1 the general formula of the relation between dimensions and angular momenta are given. It is pointed out that the solutions for all odd-dimensional problems are related to 3-dimensional ones, while those for even-dimensional problems are related to 2-dimensional solutions. In section 2, some examples are presented to show how this formula can be applied to solving Schroedinger equation for radially symmetrical potentials at higher dimensions when the solutions for 2- and 3-dimension are known. In Appendix a brief summary of the angular momentum in  $N$ -dimensional space is given.

## 1. Relation between Dimension and Angular Momentum

Consider an N-dimensional Hamiltonian

$$H = T + V(q) \quad (1.1)$$

where

$$q = (q_1, q_2, \dots, q_N) \quad (1.2)$$

and

$$T = -\frac{1}{2} \sum_{i=1}^N \partial^2 / \partial q_i^2 = -\frac{1}{2} \nabla^2. \quad (1.3)$$

The corresponding Schroedinger equation is

$$H\Psi(q) = E\Psi(q). \quad (1.4)$$

For a radially symmetric potential

$$V(q) = V(r), \quad (1.5)$$

the radial part of the wave function can be solved as a one dimensional problem by expressing Cartesian coordinates  $q_1, q_2, \dots, q_N$  in terms of the radial variable  $r$  and  $(N - 1)$  angular variables[2,4]

$$\theta_1, \theta_2, \dots, \theta_{N-2} \text{ and } \theta_{N-1} \quad (1.6)$$

through

$$\begin{aligned} q_1 &= r \cos \theta_1, & q_2 &= r \sin \theta_1 \cos \theta_2, \\ q_3 &= r \sin \theta_1 \sin \theta_2 \cos \theta_3, & \dots, \\ q_{N-1} &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{N-2} \cos \theta_{N-1} \end{aligned} \quad (1.7)$$

and

$$q_N = r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{N-2} \sin \theta_{N-1}$$

with

$$0 \leq \theta_i < \pi \text{ for } i = 1, 2, \dots, N - 2$$

and

$$0 \leq \theta_{N-1} \leq 2\pi. \quad (1.8)$$

The Laplacian operator can be expressed as[4]

$$\nabla^2 = \frac{1}{r^{2k}} \frac{\partial}{\partial r} (r^{2k} \frac{\partial}{\partial r}) - \frac{1}{r^2} \mathcal{L}^2(N-1) \quad (1.9)$$

where

$$k = \frac{1}{2}(N-1)$$

and  $\mathcal{L}^2(N-1)$  is the angular momentum operator. Its definition is given in the Appendix. According to (A.5) in the Appendix, the eigenvalues of each  $\mathcal{L}^2(n)$  are

$$l(l+n-1) \quad (1.10)$$

with  $l = 0, 1, 2, \dots$ . Thus, for a radially symmetric potential the wave function can be written as

$$\Psi(r, \theta_1, \theta_2, \dots, \theta_{N-1}) = \mathcal{R}(r)\Theta(\theta_1, \theta_2, \dots, \theta_{N-1}) \quad (1.11)$$

Taking  $l = l_1$  for the first equation of (A.7); i.e.,

$$\mathcal{L}^2(N-1)\Theta = l(l+N-2)\Theta, \quad (1.12)$$

correspondingly, the radial part of the Schroedinger equation for angular momentum  $l$  is

$$\left[ -\frac{1}{2} \nabla_r^2(k) + \frac{1}{2r^2} l(l+N-2) + V(r) - E \right] \mathcal{R}_k^l(r) = 0 \quad (1.13)$$

with

$$\nabla_r^2(k) = \frac{1}{r^{2k}} \frac{d}{dr} (r^{2k} \frac{d}{dr}). \quad (1.14)$$

Introducing

$$\mathcal{R}_k^l(r) = \frac{1}{r^k} \psi(r) \quad (1.15)$$

$\psi(r)$  satisfies the following equation

$$\left[ -\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2r^2} l(l+2k-1) + \frac{1}{2r^2} k(k-1) + V(r) \right] \psi(r) = E\psi(r). \quad (1.16)$$

Considering

$$l(l + 2k - 1) + k(k - 1) = (l + k)(l + k - 1) \quad (1.17)$$

we introduce

$$K = l + k \quad (1.18)$$

and find that  $\psi(r)$  is related only to the sum  $K = l + k$ , i.e.,  $\psi(r) = \psi_K(r)$ , therefore, (1.16) can be written as

$$\left[-\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2r^2} K(K - 1) + V(r)\right] \psi_K(r) = E \psi_K(r). \quad (1.19)$$

Defining

$$\nabla_r^2(K) = \frac{1}{r^{2K}} \frac{d}{dr} (r^{2K} \frac{d}{dr}) \quad (1.20)$$

and

$$\overline{N} = 2K + 1 = N + 2l, \quad (1.21)$$

we can introduce the radial wave function

$$\mathcal{R}_K(r) = \frac{1}{r^K} \psi_K(r) \quad (1.22)$$

satisfying

$$\left[-\frac{1}{2} \nabla_r^2(K) + V(r)\right] \mathcal{R}_K(r) = E \mathcal{R}_K(r). \quad (1.23)$$

This is just the  $S$ -state radial wave function for the same radially symmetric potential in  $\overline{N}$ -dimension. It is interesting to notice that for radially symmetric potential the wave functions  $\psi_K(r)$  for any angular momentum  $l$  in  $N$ -dimensional problem are the same as long as  $l + k = K$  and  $N = 2k + 1$ . For any combination  $K = k + l$  the radial wave function  $\mathcal{R}_k^l(r)$  with angular momentum  $l$  in  $N(= 2k + 1)$ -dimension can easily be obtained from this general function  $\psi_K(r)$  by

$$\mathcal{R}_k^l(r) = \frac{1}{r^k} \psi_K(r) = r^l \mathcal{R}_K(r). \quad (1.24)$$

(1.24) gives a very simple relation between radial wave functions with different angular momenta  $l$  and in different dimensions  $N(= 2k + 1)$  when  $l + k = K$ .

Based on (1.24), for the same radially symmetric potential they are related to the same  $\psi_K(r)$  solved from (1.19) or to the same  $\mathcal{R}_K(r)$  solved from (1.23).

This relation at least has two kinds of applications: If the full solutions for a potential in 2- and 3-dimension are known, such as Coulomb potential or spherical harmonic oscillator, it is easy to obtain the solutions for higher dimensions  $N$  through the relation (1.24). We will discuss this point in more details in the following. On the other hand, if it is relatively easy to obtain the solutions for  $l = 0$  in different dimensions  $N$ , either analytically or approximately, the solutions for  $l > 0$  in lower dimensions could be obtained by using (1.24). An application along this direction is presented for  $N$ -dimensional Sombrero-shaped potential in Ref.[4].

Now we discuss the first kind of application. Here it is necessary to discuss two cases: When  $N$  is odd,  $k$  is an integer. In this case, taking  $K = L + 1$  one reaches an equation in the 3-dimensional space with angular momentum  $L$

$$\left[-\frac{1}{2}\frac{d^2}{dr^2} + \frac{1}{2r^2}L(L+1) + V(r)\right]\psi_K(r) = E\psi_K(r). \quad (1.25)$$

Defining

$$\mathcal{R}^L(r) = \frac{1}{r}\psi_K(r) = r^L\mathcal{R}_K(r) \quad (1.26)$$

(1.25) is equivalent to the following equation in the 3-dimensional space:

$$\left[-\frac{1}{2}\nabla_r^2 + \frac{1}{2r^2}L(L+1) + V(r)\right]\mathcal{R}^L(r) = E\mathcal{R}^L(r) \quad (1.27)$$

with  $\nabla_r^2$  defined by

$$\nabla_r^2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right). \quad (1.28)$$

However, when  $N$  is even,  $k$  is a half integer. In this case, taking  $K = L + \frac{1}{2}$  one reaches an equation in 2-dimensional space:

$$\left[-\frac{1}{2}\frac{d^2}{dr^2} + \frac{1}{2r^2}\left(L^2 - \frac{1}{4}\right) + V(r)\right]\psi_K(r) = E\psi_K(r), \quad (1.29)$$

where  $L$  is the angular momentum in 2-dimensional space. Defining

$$\mathcal{R}^L(r) = \frac{1}{\sqrt{r}}\psi_K(r) = r^L\mathcal{R}_K(r) \quad (1.30)$$

(1.29) is equivalent to the following equation in 2-dimensional space:

$$\left[-\frac{1}{2}\nabla_r^2 + \frac{1}{2r^2}L^2 + V(r)\right] \mathcal{R}^L(r) = E \mathcal{R}^L(r) \quad (1.31)$$

with the corresponding  $\nabla_r^2$  defined by

$$\nabla_r^2 = \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right). \quad (1.32)$$

Therefore, if the solutions of (1.27) for any spherically symmetric potential  $V(r)$  in the 3-dimension case are known, the solutions  $\mathcal{R}_k^l(r)$  for the same potential in any odd dimension  $N = 2k + 1$  and at any angular momentum  $l$  can easily be derived via (1.26) and (1.24), taking  $K = k + l = L + 1$ . On the other hand, if the solutions of (1.31) for any spherically symmetric potential  $V(r)$  in the 2-dimension case are known, the solutions  $\mathcal{R}_k^l(r)$  for the same potential in any even dimension  $N = 2k + 1$  and at any angular momentum  $l$  can easily be derived via (1.30) and (1.24), taking  $K = k + l = L + \frac{1}{2}$ .

In the following some examples are given to show how the relation between dimensions and angular momenta is applied to solve high dimensional problems based on the known solutions for the same radial potentials in 2- or 3-dimensional cases.

## 2. Some Examples

### i) $N$ -dimensional Confined sphere

Consider  $N$ -dimensional confined sphere. The potential is expressed as

$$V(r) = \begin{cases} 0, & \text{for } r < a \\ \infty, & \text{for } r > a. \end{cases} \quad (2.1)$$

The radial wave function for angular momentum  $l$  is expressed by  $\mathcal{R}_k^l(r)$ , satisfying (1.13). Now we introduce  $K = k + l$  and defining

$$\psi_K(r) = r^k \mathcal{R}_k^l(r) \quad (2.2)$$

according to (1.15). From (1.19) we know that  $\psi_K(r)$  satisfies the following equation inside the confined sphere:

$$\left[-\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2r^2} K(K-1)\right] \psi_K(r) = E \psi_K(r) \quad (2.3)$$

and outside the sphere the wave function is zero.

For odd dimensions  $N = 2k + 1$ , based on (1.26), defining the radial wave function  $\mathcal{R}^L(r) = \frac{1}{r} \psi_K(r)$  and introducing angular momentum  $L = K - 1$  in 3-dimensional space,  $\psi_K(r)$  also satisfies the corresponding equation inside the confined sphere in 3-dimension:

$$\left[-\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2r^2} L(L+1)\right] \psi_K(r) = E \psi_K(r) \quad (2.4)$$

and outside the sphere it is zero. Introducing

$$q^2 = 2E \quad (2.5)$$

we have

$$\psi_K'' + \left[q^2 - \frac{1}{r^2} L(L+1)\right] \psi_K = 0. \quad (2.6)$$

The solution of (2.6) is the well known Bessel function[5]

$$\psi_K(r) = c\sqrt{qr} J_{L+\frac{1}{2}}(qr) \quad (2.7)$$



and the corresponding radial wave function for the 3-dimensional case is

$$\mathcal{R}^L(r) = \frac{1}{r} \psi_K(r) \quad (2.8)$$

with eigenvalues

$$E_{n_r, L} = \frac{1}{2} q_{n_r, L}^2, \quad (2.9)$$

where  $n_r$  is the radial quantum number.

Back to the odd  $N$ -dimensional case, remembering  $K = L + 1 = k + l$ , we have

$$\mathcal{R}_k^l(r) = \frac{1}{r^k} \psi_K(r) = \frac{1}{r^{k-1}} \mathcal{R}^L(r) = \frac{c}{r^k} \sqrt{qr} J_{k+l-\frac{1}{2}}(qr) \quad (2.10)$$

and the corresponding eigenvalues are

$$E_{n_r, k, l} = \frac{1}{2} q_{n_r, k+l-1}^2. \quad (2.11)$$

Now we turn to even dimensions  $N = 2k + 1$  with  $k$  a half integer. Based on (1.30), defining the radial wave function  $\mathcal{R}^L(r) = \frac{1}{\sqrt{r}} \psi_K(r)$  and introducing angular momentum  $L = K - \frac{1}{2}$  in 2-dimensional space,  $\psi_K(r)$  satisfies the corresponding equation inside the confined sphere in 2-dimension:

$$\left[ -\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2r^2} (L^2 - \frac{1}{4}) \right] \psi_K(r) = E \psi_K(r) \quad (2.12)$$

and outside the sphere the wave function is also zero. Applying the same  $q^2 = 2E$  as (2.5) we have

$$\psi_K'' + \left[ q^2 - \frac{1}{r^2} (L^2 - \frac{1}{4}) \right] \psi_K = 0. \quad (2.13)$$

The solution of (2.13) is

$$\psi_K(r) = c \sqrt{qr} J_L(qr) \quad (2.14)$$

and

$$\mathcal{R}^L(r) = \frac{1}{\sqrt{r}} \psi_K(r). \quad (2.15)$$

with eigenvalues

$$E_{n_r, L} = \frac{1}{2} q_{n_r, L - \frac{1}{2}}^2. \quad (2.16)$$

Back to the even  $N$ -dimensional case, remembering  $K = L + \frac{1}{2} = k + l$ , we have

$$\mathcal{R}_k^l(r) = \frac{1}{r^k} \psi_K(r) = \frac{1}{r^{k - \frac{1}{2}}} \mathcal{R}^L(r) = \frac{c}{r^k} \sqrt{qr} J_{k+l-\frac{1}{2}}(qr) \quad (2.17)$$

and the corresponding eigenvalues are

$$E_{n_r, k, l} = \frac{1}{2} q_{n_r, k+l-1}^2. \quad (2.18)$$

## ii) $N$ -dimensional Harmonic Oscillator

The potential of the  $N$ -dimensional harmonic oscillator is expressed as

$$V(r) = \frac{1}{2} \omega^2 r^2. \quad (2.19)$$

The radial wave function for angular momentum  $l$  is expressed by  $\mathcal{R}_k^l(r)$ , satisfying (1.13). As before, we introduce  $K = k + l$  and defining  $\psi_K(r) = r^k \mathcal{R}_k^l(r)$  as (2.2). From (1.19) we know that  $\psi_K(r)$  satisfies the following equation

$$\left[ -\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2r^2} K(K-1) + \frac{1}{2} \omega^2 r^2 \right] \psi_K(r) = E \psi_K(r). \quad (2.20)$$

For odd dimensions  $N = 2k + 1$ , based on (1.26), defining the radial wave function  $\mathcal{R}^L(r) = \frac{1}{r} \psi_K(r)$  and introducing angular momentum  $L = K - 1$  in 3-dimensional space,  $\psi_K(r)$  also satisfies the corresponding equation for harmonic oscillator in 3-dimension:

$$\left[ -\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2r^2} L(L+1) + \frac{1}{2} \omega^2 r^2 \right] \psi_K(r) = E \psi_K(r). \quad (2.21)$$

Defining  $q^2 = 2E$  we have

$$\psi_K'' + \left[ q^2 - \omega^2 r^2 - \frac{1}{r^2} L(L+1) \right] \psi_K = 0. \quad (2.22)$$

The solution of (2.22) is well known:

$$\psi_K(r) = r^{L+1} e^{-\frac{1}{2}\omega r^2} {}_1F_1(-n_r, L + \frac{3}{2}, \omega r^2) \quad (2.23)$$

and

$$\mathcal{R}^L(r) = \frac{1}{r} \psi_K(r) \quad (2.24)$$

with eigenvalues

$$E_{n_r, L} = \omega(2n_r + L + \frac{3}{2}). \quad (2.25)$$

${}_1F_1(-n_r, L + \frac{3}{2}, \omega r^2)$  is the confluent hypergeometric function[5] and  $n_r$  the radial quantum number.

Back to the odd  $N$ -dimensional case, remembering  $K = L + 1 = k + l$ , we have

$$\mathcal{R}_k^l(r) = \frac{1}{r^k} \psi_K(r) = \frac{1}{r^{k-1}} \mathcal{R}^L(r) = r^l e^{-\frac{1}{2}\omega r^2} {}_1F_1(-n_r, k+l+\frac{1}{2}, \omega r^2) \quad (2.26)$$

and the corresponding eigenvalues are

$$E_{n_r, k, l} = \omega(2n_r + k + l + \frac{1}{2}) \quad (2.27)$$

with  $n_r$  the radial quantum number.

Now we turn to even dimensions  $N = 2k + 1$  with  $k$  a half integer. Based on (1.30), defining the radial wave function  $\mathcal{R}^L(r) = \frac{1}{\sqrt{r}} \psi_K(r)$  and introducing angular momentum  $L = K - \frac{1}{2}$  in 2-dimensional space,  $\psi_K(r)$  satisfies the corresponding equation for harmonic oscillator in 2-dimension:

$$[-\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2r^2} (L^2 - \frac{1}{4}) + \frac{1}{2} \omega^2 r^2] \psi_K(r) = E \psi_K(r). \quad (2.28)$$

Defining  $q^2 = 2E$  we have

$$\psi_K'' + [q^2 - \omega^2 r^2 - \frac{1}{r^2} (L^2 - \frac{1}{4})] \psi_K = 0. \quad (2.29)$$

The solution is

$$\psi_K(r) = r^{L+\frac{1}{2}} e^{-\frac{1}{2}\omega r^2} {}_1F_1(-n_r, L + 1, \omega r^2) \quad (2.30)$$

and

$$\mathcal{R}^L(r) = \frac{1}{\sqrt{r}} \psi_K(r) \quad (2.31)$$

with eigenvalues

$$E_{n_r, L} = \omega(2n_r + L + 1). \quad (2.32)$$

Back to the even  $N$ -dimensional case, remembering  $K = L + \frac{1}{2} = k + l$ , we have

$$\mathcal{R}_k^l(r) = \frac{1}{r^k} \psi_K(r) = \frac{1}{r^{k-\frac{1}{2}}} \mathcal{R}^L(r) = r^l e^{-\frac{1}{2}\omega r^2} {}_1F_1(-n_r, k+l+\frac{1}{2}, \omega r^2) \quad (2.33)$$

and the expression for the corresponding eigenvalues is the same as (2.27), with  $k$  a half integer here.

### iii) $N$ -dimensional Coulomb Potential

The  $N$ -dimensional Coulomb potential is expressed as

$$V(r) = -\frac{Ze^2}{r}. \quad (2.34)$$

The radial wave function for angular momentum  $l$  is expressed by  $\mathcal{R}_k^l(r)$ , satisfying (1.13). As before, we introduce  $K = k + l$  and defining  $\psi_K(r) = r^k \mathcal{R}_k^l(r)$  according to (2.2). From (1.19) we know that  $\psi_K(r)$  satisfies the following equation

$$\left[ -\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2r^2} K(K-1) - \frac{Ze^2}{r} \right] \psi_K(r) = E \psi_K(r). \quad (2.35)$$

For odd dimensions  $N = 2k + 1$ , based on (1.26), defining the radial wave function  $\mathcal{R}^L(r) = \frac{1}{r} \psi_K(r)$  and introducing angular momentum  $L = K - 1$  in 3-dimensional space,  $\psi_K(r)$  also satisfies the corresponding equation for Coulomb potential in 3-dimension:

$$\left[ -\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2r^2} L(L+1) - \frac{Ze^2}{r} \right] \psi_K(r) = E \psi_K(r). \quad (2.36)$$

Defining  $\rho = \alpha r$ ,  $\alpha^2 = 8|E|$  and  $\lambda = \frac{2Ze^2}{\alpha} = \frac{Ze^2}{\sqrt{2|E|}}$ , the solution of (2.36) is [5]

$$\psi_K = \rho^{L+1} e^{-\frac{1}{2}\rho} \mathcal{L}_{n+L}^{2L+1}(\rho) \quad (2.37)$$

and

$$\mathcal{R}_{n,L} = \frac{1}{r} \psi_K \quad (2.38)$$

with eigenvalues

$$E_n = \frac{Z^2 e^4}{2n^2}. \quad (2.39)$$

$\mathcal{L}_{n+L}^{2L+1}(\rho)$  is the Laguerre polynomial. The parameters are fixed as

$$\lambda = n = n_r + L + 1 \quad \text{and} \quad \alpha_n = \frac{2Ze^2}{n} \quad (2.40)$$

with  $n_r$  the radial quantum number.

Back to the odd  $N$ -dimensional case, remembering  $K = L + 1 = k + l$ , we have

$$\mathcal{R}_k^l = \frac{1}{r^k} \psi = \frac{1}{r^{k-1}} \mathcal{R}^L = \rho^l e^{-\frac{1}{2}\rho} \mathcal{L}_{n_r+2(k+l)-1}^{2(k+l)-1}(\rho) \quad (2.41)$$

with the eigenvalues expressed by (2.39) and  $n = n_r + k + l$ .

Now we turn to even dimensions  $N = 2k + 1$  with  $k$  a half integer. Based on (1.30), defining the radial wave function  $\mathcal{R}^L(r) = \frac{1}{\sqrt{r}} \psi_K(r)$  and introducing angular momentum  $L = K - \frac{1}{2}$  in 2-dimensional space,  $\psi_K(r)$  satisfies the corresponding equation for Coulomb potential in 2-dimension:

$$\left[ -\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2r^2} (L^2 - \frac{1}{4}) + \frac{Ze^2}{r} \right] \psi_K(r) = E \psi_K(r). \quad (2.42)$$

As in the odd dimension case, defining  $\rho = \alpha r$ ,  $\alpha^2 = 8|E|$  and  $\lambda = \frac{2Ze^2}{\alpha} = \frac{Ze^2}{\sqrt{2|E|}}$ , the solution is

$$\psi_K = \rho^{L+\frac{1}{2}} e^{-\frac{1}{2}\rho} \mathcal{L}_{n+L-\frac{1}{2}}^{2L}(\rho) \quad (2.43)$$

and

$$\mathcal{R}_{n,L} = \frac{1}{\sqrt{r}} \psi_K. \quad (2.44)$$

The eigenvalues are still expressed by (2.39), with  $n = n_r + L + \frac{1}{2}$ .

Back to the even  $N$ -dimensional case, with  $K = L + \frac{1}{2} = k + l$ , we have

$$\mathcal{R}_k^l = \frac{1}{r^k} \psi_K = \frac{1}{r^{k-\frac{1}{2}}} \mathcal{R}_{n,L} = \rho^l e^{-\frac{1}{2}\rho} \mathcal{L}_{n_r+2(k+l)-1}^{2(k+l)-1}(\rho) \quad (2.45)$$

with the eigenvalues still expressed by (2.39) and  $n = n_r + k + l$ . All the obtained results for higher dimensions are consistent to earlier works.[1,6]

From above 3 examples it is shown clearly that all the eigenstates for radially symmetric  $N$ -dimensional potentials can be easily obtained if the solutions for the 2- and 3-dimensional problems with the same potential are known. Simply by substituting the quantum numbers in the expressions of the wave functions and eigenvalues according to the relation between the dimension and the angular momentum, all the solutions for odd dimensions is related to those of 3-dimensional case, while for the even dimensions they are related to those of the 2-dimensional case. It does not matter if the known solutions for 2- and 3-dimensions are analytical or numerical, the only necessary condition is a clear angular momentum dependence in the expressions. This method can also be applied to approximate solutions under the same condition.

## Appendix A. Angular Momentum Operator<sup>[4]</sup>

Express the Cartesian coordinates  $q_1, q_2, \dots, q_N$  in terms of the radial variable  $r$  and  $(N - 1)$  angular variables

$$\theta_1, \theta_2, \dots, \theta_{N-2} \text{ and } \theta_{N-1} \quad (A.1)$$

according to (1.7)-(1.8). Correspondingly, the line elements are

$$\begin{aligned} dr, \quad r d\theta_1, \quad r \sin \theta_1 d\theta_2, \quad r \sin \theta_1 \sin \theta_2 d\theta_3, \\ r \sin \theta_1 \sin \theta_2 \sin \theta_3 d\theta_4, \dots, \quad r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{N-2} d\theta_{N-1}. \end{aligned} \quad (A.2)$$

In the Laplacian operator given in (1.9)

$$\nabla^2 = \frac{1}{r^{2k}} \frac{\partial}{\partial r} (r^{2k} \frac{\partial}{\partial r}) - \frac{1}{r^2} \mathcal{L}^2(N-1)$$

the angular momentum operators  $\mathcal{L}^2(N-1)$  are defined as

$$\begin{aligned} \mathcal{L}^2(N-1) &= -\frac{1}{\sin^{N-2} \theta_1} \frac{\partial}{\partial \theta_1} (\sin^{N-2} \theta_1 \frac{\partial}{\partial \theta_1}) + \frac{1}{\sin^2 \theta_1} \mathcal{L}^2(N-2) \\ \mathcal{L}^2(N-2) &= -\frac{1}{\sin^{N-3} \theta_2} \frac{\partial}{\partial \theta_2} (\sin^{N-3} \theta_2 \frac{\partial}{\partial \theta_2}) + \frac{1}{\sin^2 \theta_2} \mathcal{L}^2(N-3) \\ &\dots \quad \dots \\ \mathcal{L}^2(2) &= -\frac{1}{\sin \theta_{N-2}} \frac{\partial}{\partial \theta_{N-2}} (\sin \theta_{N-2} \frac{\partial}{\partial \theta_{N-2}}) + \frac{1}{\sin^2 \theta_{N-2}} \mathcal{L}^2(1) \end{aligned} \quad (A.3)$$

and

$$\mathcal{L}^2(1) = -\frac{\partial^2}{\partial \theta_{N-1}^2}.$$

The square of the angular momentum operator on an  $n$ -sphere is  $\mathcal{L}^2(n)$ . The commutator between any two  $\mathcal{L}^2(n)$  and  $\mathcal{L}^2(m)$  is zero; i.e.,

$$[\mathcal{L}^2(n), \mathcal{L}^2(m)] = 0. \quad (A.4)$$

The eigenvalues of each  $\mathcal{L}^2(n)$  are

$$l(l + n - 1) \quad (A.5)$$

with  $l = 0, 1, 2, \dots$ . The derivation of the eigenvalues of  $\mathcal{L}^2(n)$  can be found in Appendix A of Ref.[4].

Thus, for a radially symmetric potential, the wave function can be written as

$$\psi(r, \theta_1, \theta_2, \dots, \theta_{N-1}) = \mathcal{R}(r)\Theta(\theta_1, \theta_2, \dots, \theta_{N-1}) \quad (A.6)$$

with

$$\begin{aligned} \mathcal{L}^2(N-1)\Theta &= l_1(l_1 + N - 2)\Theta, \\ \mathcal{L}^2(N-2)\Theta &= l_2(l_2 + N - 3)\Theta, \\ &\dots \end{aligned} \quad (A.7)$$

$$\mathcal{L}^2(2)\Theta = l_{N-2}(l_{N-2} + 1)\Theta$$

and

$$\mathcal{L}^2(1)\Theta = l_{N-1}^2\Theta.$$

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